

Self-sustained amplification of disturbances in pipe Poiseuille flow

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Abstract – The nonlinear development of disturbances in pipe Poiseuille flow is studied with a low-dimensional model. The basic system from which the model is derived governs disturbances closely related to the radial velocity and radial vorticity disturbances. The analysis is restricted to the interaction of the two first harmonics of streamwise elongated disturbances since they are the most transiently amplified ones in linear theory. In the resulting dynamical system a nonlinear feedback from the normal vorticity disturbance (which is transiently amplified according to linear theory) to the radial velocity disturbance is present. Above a threshold of the initial amplitude, the feedback leads to a self-sustained amplification of the disturbances continuing for all times. © Elsevier, Paris

1. Introduction

In 1883 Osborne Reynolds made the remarkable discovery of the transition from laminar to turbulent flow in a circular pipe [1]. The transition occurred if the Reynolds number (R) exceeded a certain value. To explain the transition has since been one of the major efforts in fluid mechanics research. In a number of subsequent experiments by, for example, Wygnanski and Champagne [2], Wygnanski et al. [3] and Fox et al. [4], the lowest R (based on the centerline velocity and the pipe radius) for transition is found to be about 2000–2200.

For a fully developed parabolic mean velocity profile, no theoretical explanation of the transition was ever found in classical stability analysis of the linearized eigenvalue problem. In studies of axisymmetric disturbances by Pekeris [5], Corcos and Sellars [6] and Davey and Drazin [7], only damped modes were found. Herron [8] finally derived a formal proof of stability in the axisymmetric case. For angular dependent disturbances no formal proof of stability exists but from extensive numerical studies by, for example, Lessen et al. [9] and Salwen et al. [10] also angular dependent disturbances are believed to be stable. For finite amplitude disturbances the axisymmetric case was studied by Davey and Nguyen [11], Itoh [12] and Davey [13] but the lack of a neutral stability curve has made the use of expansion techniques difficult and not without controversy. In numerical simulations of finite-amplitude axisymmetric disturbances [14] however, only decaying solutions were found.

The failure of traditional stability analysis to predict the instability has focused interest on alternative mechanisms not based on the classical concept of stability. Such a mechanism is the transient growth mechanism. This mechanism admits a large amplification of small disturbances because of the non-normal properties of the equations governing the disturbances, i.e. the eigenmodes are not orthogonal. In planar flow cases the transient mechanism has been studied by, for example, Farrell [15], Gustavsson [16], Butler and Farrell [17] and Reddy et al. [18].

In simulations of the nonlinear angular dependent initial value problem in pipe flow, Boberg and Brosa [19] observed a transient behaviour of the solution. By analysis of the numerical solutions it was revealed that linear terms are responsible for the transient amplification. A linear model consisting of a decaying solution and a

transiently amplified solution was suggested to provide amplification despite the linear stability. Bergström [20] analytically considered transient growth of small angular dependent disturbances without a streamwise dependence (streamwise wavenumber $\alpha = 0$) and found an amplification of the streamwise disturbance energy density proportional to R^2 . For the general case of disturbances with an arbitrary streamwise dependence, the optimal transient growth was studied by Bergström [21] and Schmid and Henningson [22]. The disturbances of the largest amplification were found to be those without a streamwise dependence and with an azimuthal periodicity of unity.

In experiments on transient growth in pipe flow by Bergström [23,24], the transient behaviour of the streamwise disturbance velocity at subcritical Reynolds number was verified. The propagation velocities of the disturbances also agreed with the propagation velocities found analytically by Bergström [21] for the most amplified disturbances.

Although the linear transient mechanism theoretically gives a large disturbance amplification, the disturbances eventually decay. Therefore, if the transient growth mechanism is a prerequisite for transition some nonlinear mechanism must be involved in addition to it. Boberg and Brosa [19] suggested that the role of nonlinearity is to support the transient mechanism indicated in their simulations by transforming part of the amplified solution into the decaying solution and thereby continually restarting the linear transient process. Subsequently, a number of low-dimensional ad hoc models for shear flows have been presented, where a linear non-normal operator combined with nonlinearity results in a sustained disturbance amplification above some threshold of the initial amplitude. There has been a debate whether nonlinear feedback occurs for a real flow case or not and different routes to transition have been suggested in the models. However, Baggett and Trefethen [25] have compared the different models and they assert that the different ideas correspond more closely than it may at first appear. The question of nonlinear feedback will be discussed later in this paper.

In the present paper, a nonlinear low-dimensional model is derived from the equations governing the disturbance development in pipe Poiseuille flow. The motive of the study is to investigate whether a low-dimensional model of disturbances in pipe Poiseuille flow admits sustained amplification. In Section 2 the governing equations for three-dimensional disturbances in pipe Poiseuille flow are given and the model is derived. By exclusively considering the nonlinear interaction of the first modes of the two first harmonics, a dynamical system of eight equations governing the time-development of the disturbances results. In Section 3 the time development of the disturbances and threshold amplitudes for sustained amplification are presented. Finally, in Section 4, the results are discussed and summarised.

2. The governing equations

Consider incompressible viscous flow through a circular pipe. In a cylindrical (x, r, θ) coordinate system with a laminar mean flow $U = 1 - r^2$ in the streamwise direction (x), the development of three-dimensional velocity disturbances (u, v, w) and the associated pressure p is governed by the Navier–Stokes equations

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + U'v + N_1 = -\frac{\partial p}{\partial x} + \frac{1}{R} \nabla^2 u, \quad (1a)$$

$$\frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} + N_2 = -\frac{\partial p}{\partial r} + \frac{1}{R} \left(\nabla^2 v - \frac{v}{r^2} - \frac{2}{r^2} \frac{\partial w}{\partial \theta} \right), \quad (1b)$$

$$\frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} + N_3 = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{1}{R} \left(\nabla^2 w + \frac{2}{r^2} \frac{\partial v}{\partial \theta} - \frac{w}{r^2} \right) \quad (1c)$$

and continuity

$$\frac{\partial u}{\partial x} + \frac{1}{r} \frac{\partial(rv)}{\partial r} + \frac{1}{r} \frac{\partial w}{\partial \theta} = 0 \quad (2)$$

where the nonlinear terms N_1 , N_2 and N_3 are defined by

$$N_1 = v \frac{\partial u}{\partial r} + \frac{w}{r} \frac{\partial u}{\partial \theta} + u \frac{\partial u}{\partial x}, \quad (3a)$$

$$N_2 = v \frac{\partial v}{\partial r} + \frac{w}{r} \frac{\partial v}{\partial \theta} + u \frac{\partial v}{\partial x} - \frac{w^2}{r}, \quad (3b)$$

$$N_3 = v \frac{\partial w}{\partial r} + \frac{w}{r} \frac{\partial w}{\partial \theta} + u \frac{\partial w}{\partial x} + \frac{vw}{r}. \quad (3c)$$

In Eqs (1)–(3), all quantities are made dimensionless by the pipe radius and the mean flow centerline velocity, R is the Reynolds number, ∇^2 is the cylindrical Laplacian and the prime indicates differentiation with respect to r . Since the flow is periodic in the θ direction and is assumed to be periodic in the x direction, the disturbances can be expanded in components according to

$$\begin{aligned} (u, v, w, p)^T &= \frac{1}{2} \sum_m (\bar{u}_m(t, r), \bar{v}_m(t, r), \bar{w}_m(t, r), \bar{p}_m(t, r))^T e^{im(\alpha x + \theta)} \\ &+ \frac{1}{2} \sum_m (\bar{u}_m^*(t, r), \bar{v}_m^*(t, r), \bar{w}_m^*(t, r), \bar{p}_m^*(t, r))^T e^{-im(\alpha x + \theta)} \end{aligned} \quad (4)$$

where the $*$ denotes the complex conjugate, α is the streamwise wavenumber and m (integer) indicates the periodicity. A more general form of ansatz for the streamwise and azimuthal dependence has been $\sum_m \sum_n e^{i(m\alpha x + n\theta)}$. However, for the purpose of this work, the form (4) will be more convenient. Also, in (4) $m \neq 0$ is assumed, since only such disturbances will be considered here. By eliminating the pressure in the original equations, the problem can be described by two equations which govern quantities related to the radial velocity disturbance and the radial vorticity disturbance (see, e.g. [22,26]). By introducing

$$\Omega_m(t, r) = \frac{\alpha_m r \bar{w}_m - m \bar{u}_m}{m R k_m^2 r^2} \quad \text{and} \quad \Phi_m(t, r) = -ir \bar{v}_m \quad (5)$$

where

$$\alpha_m = m\alpha \quad \text{and} \quad k_m^2 = \alpha_m^2 + \frac{m^2}{r^2}, \quad (6)$$

the nonlinear versions of the equations governing Φ_m and Ω_m are given by

$$\begin{aligned} &\left(R \frac{\partial}{\partial t} + i\alpha_m R U \right) T \Phi_m - \frac{i\alpha_m R}{r} \left(\frac{U'}{k_m^2 r} \right)' \Phi_m \\ &= -2\alpha_m R m^2 T \Omega_m + T (k_m^2 r^2 T) \Phi_m - \frac{R}{\alpha_m r} \left[i\alpha_m N_2^{(m)} - \frac{\partial N_1^{(m)}}{\partial r} - m \frac{\partial}{\partial r} \left(\frac{\alpha_m r N_3^{(m)} - m N_1^{(m)}}{k_m^2 r^2} \right) \right] \end{aligned} \quad (7a)$$

and

$$k_m^2 r^2 \left(R \frac{\partial}{\partial t} + i\alpha_m R U \right) \Omega_m - \frac{iU'}{r} \Phi_m = k_m^2 r^2 S \Omega_m + 2 \frac{\alpha_m}{R} T \Phi_m - \left[\frac{\alpha_m r N_3^{(m)} - m N_1^{(m)}}{m} \right], \quad (7b)$$

respectively, where the superscript $^{(m)}$ indicates nonlinear terms of periodicity m and where the operators T and S are defined by

$$T = \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{1}{k_m^2 r} \frac{\partial}{\partial r} \right) - \frac{1}{r^2} \quad \text{and} \quad S = \frac{1}{r^2 k_m^2} \frac{1}{r} \frac{\partial}{\partial r} \left(k_m^2 r^3 \frac{\partial}{\partial r} \right) - k_m^2. \quad (8)$$

The problem is now completely described by Ω_m and Φ_m since \bar{u}_m and \bar{w}_m in the nonlinear terms are expressed in Ω_m and Φ_m through the relations

$$\bar{u}_m = -\frac{\alpha_m}{k_m^2 r} \frac{\partial \Phi_m}{\partial r} - m^2 R \Omega_m \quad \text{and} \quad \bar{w}_m = -\frac{m}{k_m^2 r^2} \frac{\partial \Phi_m}{\partial r} + \alpha_m m r R \Omega_m. \quad (9)$$

The boundary conditions to be satisfied by Ω_m and Φ_m are

- at $r = 1$:

$$\Phi_m = \Omega_m = \frac{\partial \Phi_m}{\partial r} = 0; \quad (10a)$$

- at $r = 0$:

$$\begin{aligned} \Phi_m = \Omega_m = 0, \quad \frac{\partial \Phi_m}{\partial r} \text{ finite} \quad & \text{for } m = 1, \\ \Phi_m = \frac{\partial \Phi_m}{\partial r} = \Omega_m = 0 \quad & \text{for } m \geq 2. \end{aligned} \quad (10b)$$

From linear theory it is known that $\alpha = 0$ disturbances exhibit the largest transient amplification owing to non-normality. However, for pure $\alpha = 0$ disturbances, the disturbance which is transiently amplified (according to linear theory) does not affect the forcing disturbance through nonlinearity (see [27]). The present study of the nonlinear development of the disturbances will therefore be focused on disturbances highly (but not infinitely) elongated in the streamwise direction. Specifically, the linear parts of the governing equations are approximated by their $\alpha = 0$ form, while in the nonlinear terms, α will satisfy $0 < \alpha \ll 1$. The use of the $\alpha = 0$ version of the linear part of the equations is justified since the transient linear disturbance development for $\alpha \ll 1$ is well represented by the development of $\alpha = 0$ disturbances. The linear α -terms modify only slightly the linear transient growth if α is small while the nonlinear α -terms are crucial for a possible nonlinear feedback and sustained amplification. Typically, the magnitude of α will be of some hundredth-parts which then makes $\alpha R \gg 1$ for R values of practical interest for transition in pipe Poiseuille flow (i.e. $R \gtrsim 2000$).

The work will from now be focused on a low-dimensional model for the disturbance development and only the nonlinear interaction of the two first components, i.e. $m = 1$ and $m = 2$, which are the most amplified ones in linear theory for small α , will therefore be considered. Each component can in general be represented as an infinite set of modes with time-dependent coefficients. Here, the eigenmodes associated with the linear and homogeneous $\alpha = 0$ forms of (7a)–(7b) will be exploited. Also, for each m just the first mode of Φ_m and the first mode of Ω_m will be included in the final model. It should be pointed out that for $\alpha = 0$ disturbances, the largest contribution to the transient growth comes from the first (and least damped) modes of Φ_m and Ω_m . For larger α , the major contribution comes from the modes at the intersection of the eigenvalue branches (and not from the least damped modes). The dominance of the first modes for $\alpha = 0$ will be further substantiated in Section 3. It should also be emphasized that it is the combination of the modes of Φ_m and Ω_m that causes the transient amplification.

For $\alpha = 0$, the eigenmodes are analytically accessible and the first modes of Φ_m and Ω_m are given by

$$\phi_m(r) = J_m(j_{m+1,1}r) - J_m(j_{m+1,1})r^m \quad (11a)$$

and

$$\omega_m(r) = J_m(j_{m,1}r), \quad (11b)$$

respectively, where $j_{m,1}$ and $j_{m+1,1}$ are the first zeros of the Bessel functions J_m and J_{m+1} , respectively. In a first mode model of $m = 1$ and $m = 2$ components, Ω_m , Φ_m , Ω_m^* and Φ_m^* are then decomposed into time dependent and radially dependent parts according to

$$\begin{aligned} \Phi_m &= a_m(t)\phi_m(r), & \Phi_m^* &= a_m^*(t)\phi_m(r) \\ \Omega_m &= b_m(t)\omega_m(r), & \Omega_m^* &= b_m^*(t)\omega_m(r) \end{aligned} \quad (m = 1, 2). \quad (12)$$

With (12) inserted into Eqs (7a)–(7b) and by exclusively considering $e^{i(\alpha x + \theta)}$ and $e^{2i(\alpha x + \theta)}$ terms, four equations are obtained. The equations governing Φ_m and Ω_m are then multiplied by $J_m(j_{m+1,1}r)$ and $J_m(j_{m,1}r)$, respectively, and then integrated $\int_0^1 \dots r dr$. By separating the real and imaginary parts of the four radially integrated equations, a system of eight equations governing the real and imaginary parts of a_1, a_2, b_1 and b_2 (denoted $a_{1r}, a_{1i}, a_{2r}, a_{2i}, b_{1r}, b_{1i}, b_{2r}$ and b_{2i}) results. After some manipulations, the dynamical system governing the time-development of the disturbances is formally given by

$$\begin{aligned} \frac{da_{1r}}{dt} &= -\frac{j_{2,1}^2}{R}a_{1r} - \alpha^2 R^2(b_{1i}b_{2r} - b_{2i}b_{1r})\xi_1 - R(a_{1i}b_{2r} - a_{1r}b_{2i})\xi_2 \\ &\quad - R(a_{2r}b_{1i} - a_{2i}b_{1r})\xi_3 - (a_{1i}a_{2r} - a_{1r}a_{2i})\xi_4, \end{aligned} \quad (13a)$$

$$\begin{aligned} \frac{da_{1i}}{dt} &= -\frac{j_{2,1}^2}{R}a_{1i} - \alpha^2 R^2(b_{1r}b_{2r} + b_{1i}b_{2i})\xi_1 - R(a_{1r}b_{2r} + a_{1i}b_{2i})\xi_2 \\ &\quad - R(a_{2r}b_{1r} + a_{2i}b_{1i})\xi_3 - (a_{1r}a_{2r} + a_{1i}a_{2i})\xi_4, \end{aligned} \quad (13b)$$

$$\frac{da_{2r}}{dt} = -\frac{j_{3,1}^2}{R}a_{2r} + 2\alpha^2 R^2 b_{1r}b_{1i}\xi_5 + R(a_{1r}b_{1i} + a_{1i}b_{1r})\xi_6 + 2a_{1r}a_{1i}\xi_7, \quad (13c)$$

$$\frac{da_{2i}}{dt} = -\frac{j_{3,1}^2}{R}a_{2i} - \alpha^2 R^2(b_{1r}b_{1r} - b_{1i}b_{1i})\xi_5 - R(a_{1r}b_{1r} - a_{1i}b_{1i})\xi_6 - (a_{1r}a_{1r} - a_{1i}a_{1i})\xi_7, \quad (13d)$$

$$\frac{db_{1r}}{dt} = -\frac{j_{1,1}^2}{R}b_{1r} - \frac{2}{R}\ell_1 a_{1i} + (a_{1i}b_{2r} - a_{1r}b_{2i})\xi_8 + (a_{2r}b_{1i} - a_{2i}b_{1r})\xi_9, \quad (13e)$$

$$\frac{db_{1i}}{dt} = -\frac{j_{1,1}^2}{R}b_{1i} + \frac{2}{R}\ell_1 a_{1r} + (a_{1r}b_{2r} + a_{1i}b_{2i})\xi_8 + (a_{2r}b_{1r} + a_{2i}b_{1i})\xi_9, \quad (13f)$$

$$\frac{db_{2r}}{dt} = -\frac{j_{2,1}^2}{R}b_{2r} - \frac{2}{R}\ell_2 a_{2i} - (a_{1r}b_{1i} + a_{1i}b_{1r})\xi_{10}, \quad (13g)$$

$$\frac{db_{2i}}{dt} = -\frac{j_{2,1}^2}{R}b_{2i} + \frac{2}{R}\ell_2 a_{2r} + (a_{1r}b_{1r} - a_{1i}b_{1i})\xi_{10} \quad (13h)$$

where the coefficients ℓ_1, ℓ_2 and ξ_1 – ξ_{10} are radially integrated terms for which the numerical values are given in the Appendix. The terms containing ℓ_1 and ℓ_2 are the forcing terms causing the transient amplification of Ω in linear theory. Except for ξ_1 and ξ_5 all other ξ 's depend on α since α cannot be factored out from them. (The reason is that α appears among other things as one of the terms in the denominator and must be included with a given value in the numerical radial integration). In the model (13), the presence of b -terms in the a -equations implies the existence of a possible nonlinear effect from b to a . Therefore, a nonlinear influence from the transiently amplified Ω (according to linear theory) to Φ may be possible.

In view of the question whether or not nonlinear feedback occurs, the presence of nonlinear Ω -terms in the Φ -equations deserves some comment. In plane Poiseuille flow, Benney and Gustavsson [28] and Shanthini [29] have shown that the nonlinear terms of the equation governing the normal velocity (v) do not contain the transiently amplified normal vorticity (η) if oblique disturbances of the form $e^{im(\alpha x + \beta z)}$ are considered (α and β are streamwise and spanwise wavenumbers). If instead for example a pair of oblique disturbances of the form $e^{i(\alpha x \pm \beta z)}$ are studied the situation becomes different and nonlinear terms containing η occur in the equation governing v and thus there is a potential for a nonlinear feedback from η to v . In a turbulent flow case, Waleffe et al. [30] investigated the different types of nonlinear terms for the particular case of a direct resonance between v and η . They hypothesized that the dominant regeneration process is a result of the nonlinear self-interaction of the v -terms and that the nonlinear η -terms are small. However, the direct resonance involves higher modes of larger damping which suggests that only a small growth of η is possible compared to the amplifications obtained from the general transient growth mechanism. For example, in the case of disturbances on a laminar mean flow, Gustavsson [16] reported an induced energy density amplification of 177.9 times for $R = 1000$ owing to the general transient mechanism. The corresponding value for a direct resonance was 16.5 times.

In pipe Poiseuille flow, the form of (13) implies that it is sufficient to exploit the ansatz $e^{im(\alpha x + \theta)}$ to obtain nonlinear Ω -terms in the Φ -equations and it is not necessary to use the more general ansatz $\sum_m \sum_n e^{i(m\alpha x + n\theta)}$. However, just the presence of nonlinear Ω -terms does not automatically imply that the feedback is of importance. The subsequent analysis has to reveal whether the feedback in (13) effects the development, i.e. does it counteract or support the establishment of a sustained amplification in pipe Poiseuille flow? Naturally, the model (13) is a very simplified model where for example no generation of higher components is included. It cannot be expected that a very complex phenomenon like laminar-turbulent transition can be fully described by a model like (13). However, in spite of the simplicity, if the full basic equations governing the stability of pipe Poiseuille flow allow a sustained disturbance amplification, the transition may also be indicated by a low-dimensional model derived from the basic equations.

The system (13) is solved numerically by a 4th and 5th order Runge–Kutta–Fehlberg integration method. The resulting disturbance development is characterized by the energy density of the disturbances defined by

$$E_m = \pi \int_0^1 \left[\frac{\Phi'_m \Phi_m^*}{k_m^2 r^2} + \frac{\Phi_m \Phi_m^*}{r^2} + R^2 m^2 k_m^2 r^2 \Omega_m \Omega_m^* \right] r dr \quad (14)$$

normalized by the initial energy density denoted E_{m0} . The initial disturbance is defined by giving a_{1r} and a_{1i} equal values (all the others are zero initially) and the parameter $\varepsilon = a_{1r} = a_{1i}$ is used to define the initial disturbance strength.

3. Results

3.1. The time development of the disturbances

In figure 1 the development of E_1/E_{10} is presented for $R = 4000$, $\alpha = 0.03$ and $\varepsilon = 0.004$ (dashed dotted), $\varepsilon = 0.0079$ (dashed) and $\varepsilon = 0.008$ (solid). For $\varepsilon = 0.004$, the development is as in the linear case, i.e. the disturbance is transiently amplified to a peak and then decays. The peak occurs at $t = 198$ and the maximum amplification is 1075. In spite of the few modes included in the model (13), these values coincide well with the values found in linear studies of $\alpha = 0$ disturbances where the peak position occurs at $t/R \simeq 0.049$ and the optimal amplification for $R = 1000$ is $E_1/E_{10} \simeq 72$. With $R = 4000$, the corresponding values are $t = 196$ and $E_1/E_{10} = 1152$ (since the total energy density amplification roughly scales with R^2 for small α). In the

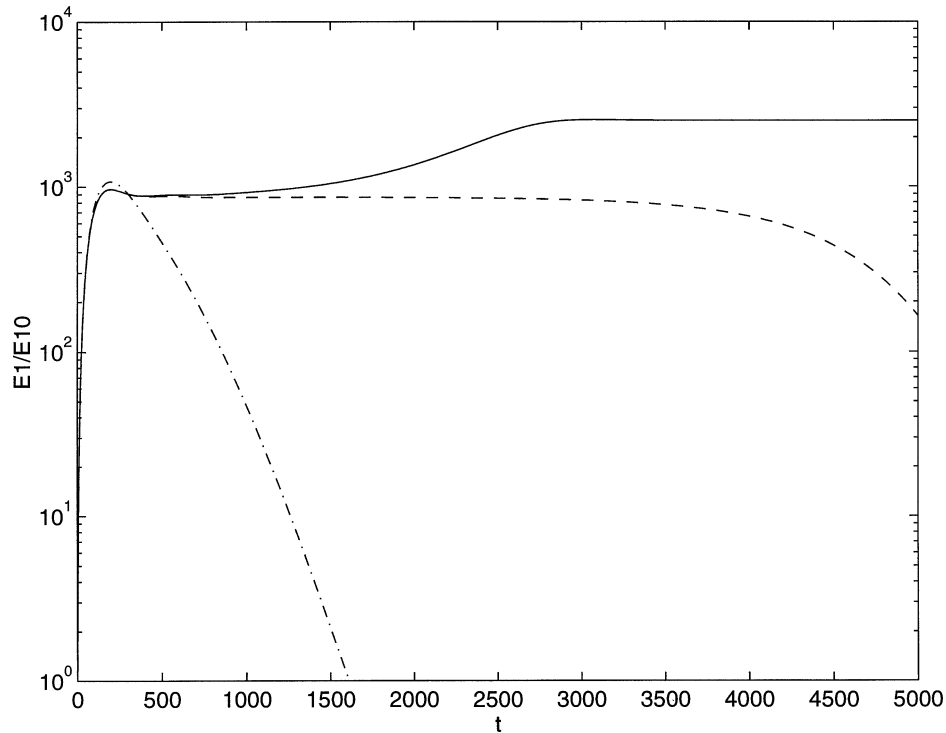


Figure 1. Development of E_1/E_{10} for $R = 4000$, $\alpha = 0.03$, $\varepsilon = 0.004$ (dashed dotted), $\varepsilon = 0.0079$ (dashed) and $\varepsilon = 0.008$ (solid).

case $\varepsilon = 0.0079$, the disturbance does not start to decay immediately after the linear peak is reached as in the previous case. Here, nonlinearity has an influence and the amplification is maintained for a substantially longer period until the decay rate finally increases and the disturbance rapidly decays. With $\varepsilon = 0.008$, the disturbance behaves at first as in the previous case until $t \simeq 700$. Then it becomes further amplified to about 2.3 times the linear maximum amplification and a sustained state of amplification continuing for all times becomes established. Although the model (13) solely describes the nonlinear development of disturbances highly elongated in the streamwise direction, the results indicate that the linear transient growth combined with nonlinearity is capable of causing a sustained amplification of disturbances in pipe Poiseuille flow. For the same values of parameters as in the sustained case presented in *figure 1*, the phase-plane development of b_{1r} versus b_{1i} from (13) is presented in *figure 2*. The calculations are performed for $0 \leq t \leq 100 \cdot 10^3$ and the star indicates the initial point. Initially, owing to the choice of initial disturbance, b_{1r} and b_{1i} are zero. After a while, the trajectory follows a fixed track and the sustained state of amplification is represented by a limit cycle of circular shape. (The same behaviour is obtained for other pairs of (13)). As a consequence, since for example E_1/E_{10} depends on $a_{1r}^2 + a_{1i}^2$ and $b_{1r}^2 + b_{1i}^2$, the energy density becomes constant although the individual components of (13) oscillate. Although it is not formally proven that the sustained amplification continues for all times, the result showing a sustained limit cycle for t up to $100 \cdot 10^3$ makes this plausible.

In *figure 1*, for all three curves, the initial growing phase up to the linear peak is almost identical. However, the amount of amplification (at the linear peak) decreases somewhat as ε decreases. The decrease of the transient (linear) amplification by nonlinearity has also been reported by O'Sullivan and Breuer [31] and by Bergström [27]. For $m = 2$, a similar development as presented for $m = 1$ is obtained. The energy density of the $m = 2$ component however is zero initially and the initial growth is induced by the nonlinear interaction of the $m = 1$ components.

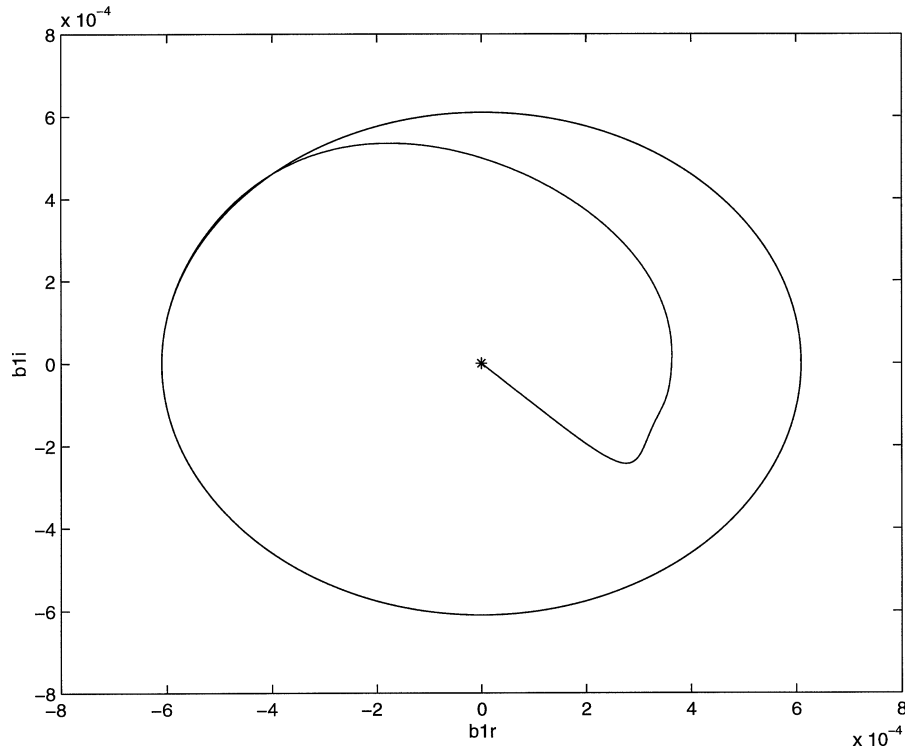


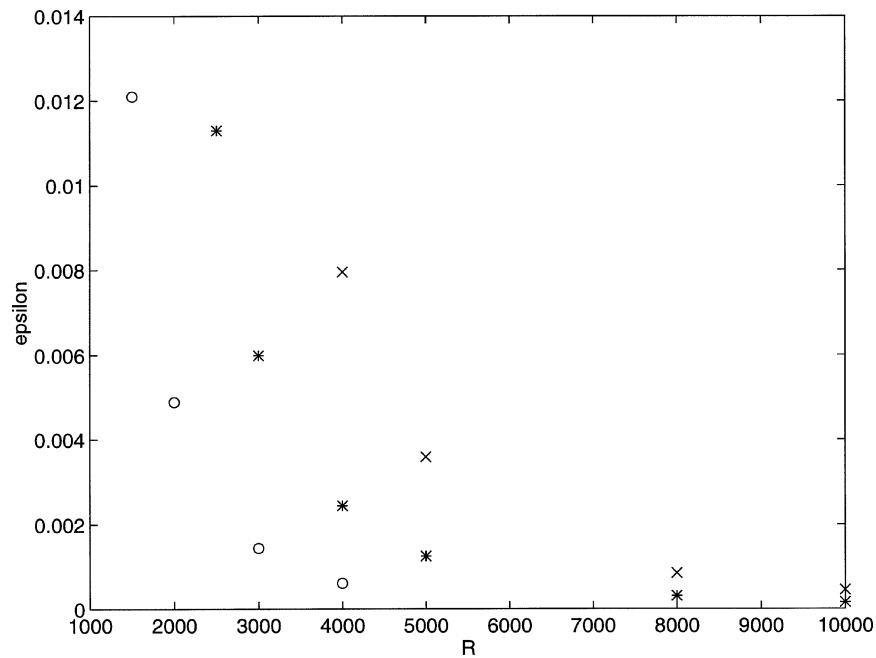
Figure 2. Phase-plane development of b_{1r} versus b_{1i} for $\varepsilon = 0.008$, $\alpha = 0.03$, $R = 4000$ and $0 \leq t \leq 100 \cdot 10^3$.

3.2. Threshold amplitudes of ε for a sustained amplification

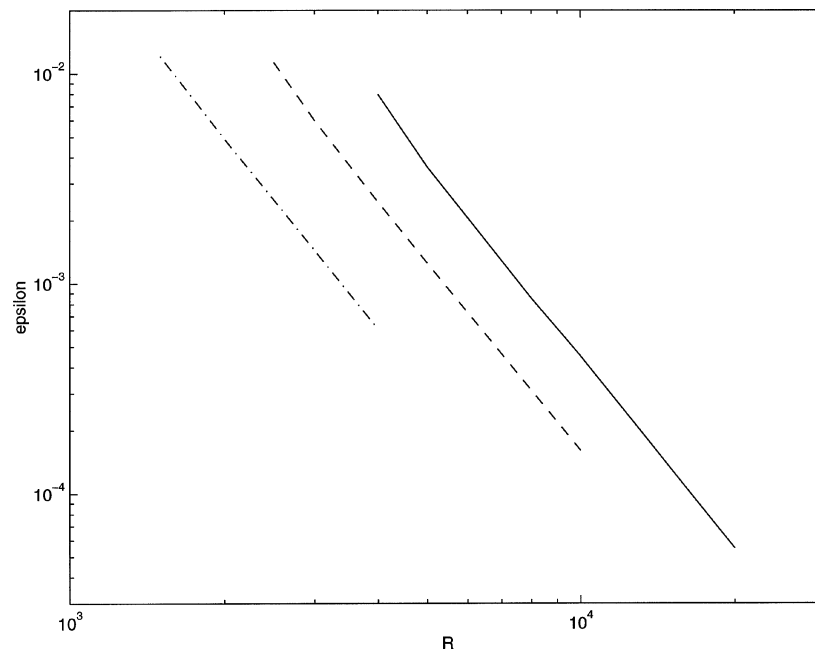
From numerical tests, the threshold (i.e. the lowest value) of ε for a sustained amplification versus the corresponding R is presented in *figure 3a* for $\alpha = 0.03$ (cross), $\alpha = 0.05$ (star) and $\alpha = 0.1$ (circle). The curves are displaced towards larger R as α decreases and the threshold amplitude for sustained amplification begins to be relatively high for values of R below some thousands. In addition, for $\alpha \ll 1$ and lower R (but still $\alpha R \gg 1$, for example, $\alpha = 0.03$ and $R = 1000$), it can be pointed out that no sustained state has been obtained. For larger α and lower R , sustained amplification can be achieved but the requirement that $\alpha \ll 1$ in the derivation of the model becomes violated. In *figure 3b* the threshold amplitude of ε versus the corresponding value of R is presented in a logarithmic diagram for $\alpha = 0.03$ (solid), $\alpha = 0.05$ (dashed) and $\alpha = 0.1$ (dashed dotted). With an assumed threshold relation of the form $\varepsilon = \text{constant} R^g$, all curves in *figure 3b* indicate unambiguously that $g = -3$. In the next section, the indicated value of g will be explained by a heuristic consideration of Eq. (13). It can be mentioned that in some tests with ε well beyond the lowest ε for sustained amplification, the solution may decay for a narrow interval of ε and then again become sustained outside the interval. This observation has been made close to the lowest R for which sustained solutions have been achieved for a certain α .

4. Discussion and conclusion

The nonlinear development of disturbances related to the radial velocity and radial vorticity disturbances in pipe Poiseuille flow has been studied with a low-dimensional model. The model concerns disturbances highly elongated in the streamwise direction (i.e. $\alpha \ll 1$) since they are the most amplified ones in linear



(a)

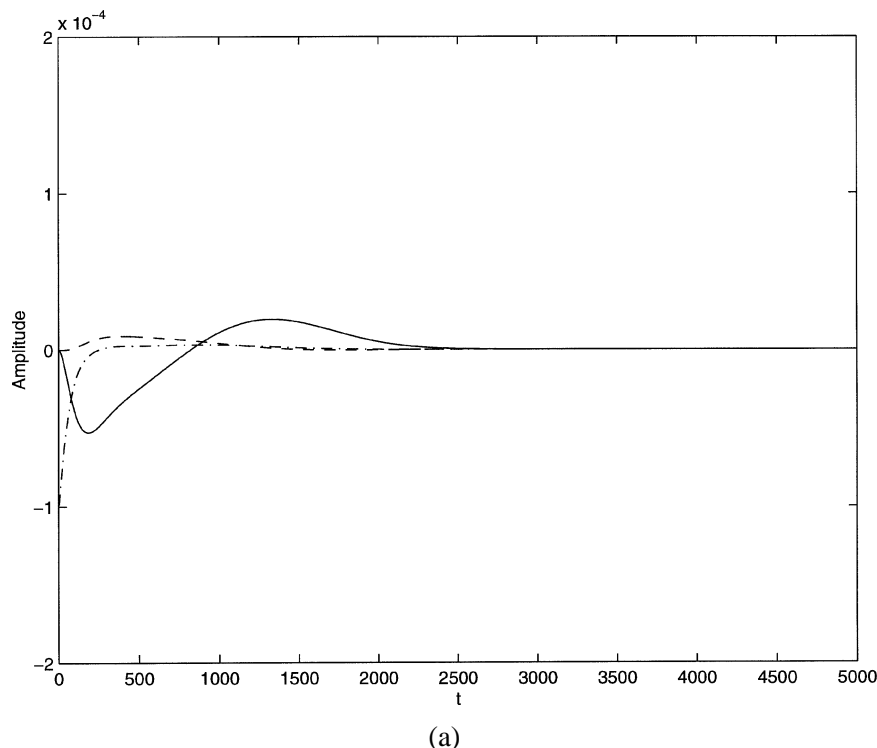


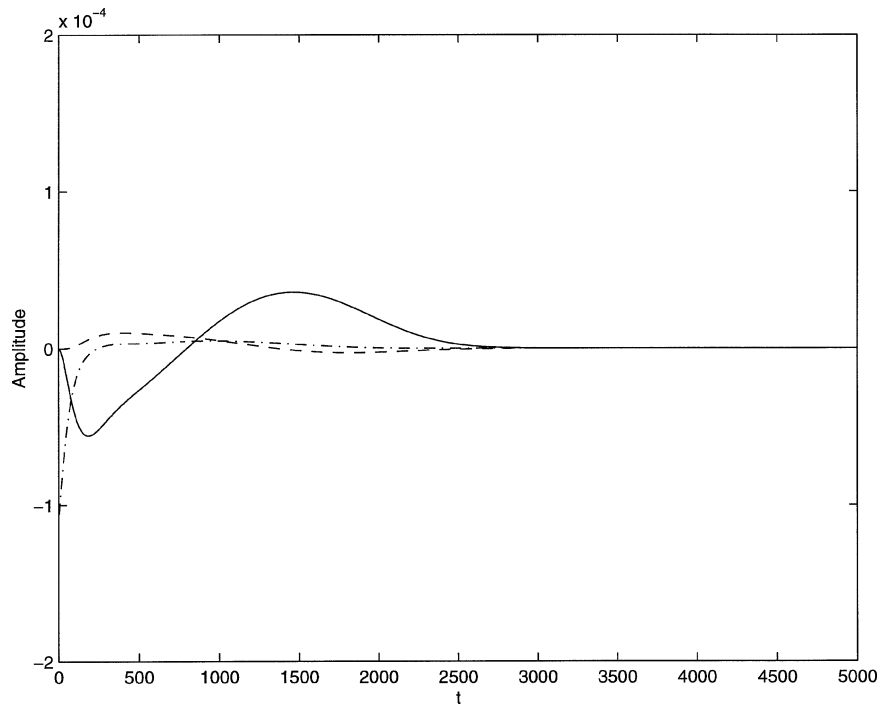
(b)

Figure 3. (a) Numerically obtained threshold amplitudes of ϵ for sustained amplification versus the Reynolds number R . $\alpha = 0.03$ (cross), $\alpha = 0.05$ (star) and $\alpha = 0.1$ (circle). (b) Logarithmic diagram of the threshold amplitude of ϵ versus the corresponding value of R for $\alpha = 0.03$ (solid), $\alpha = 0.05$ (dashed) and $\alpha = 0.1$ (dashed dotted).

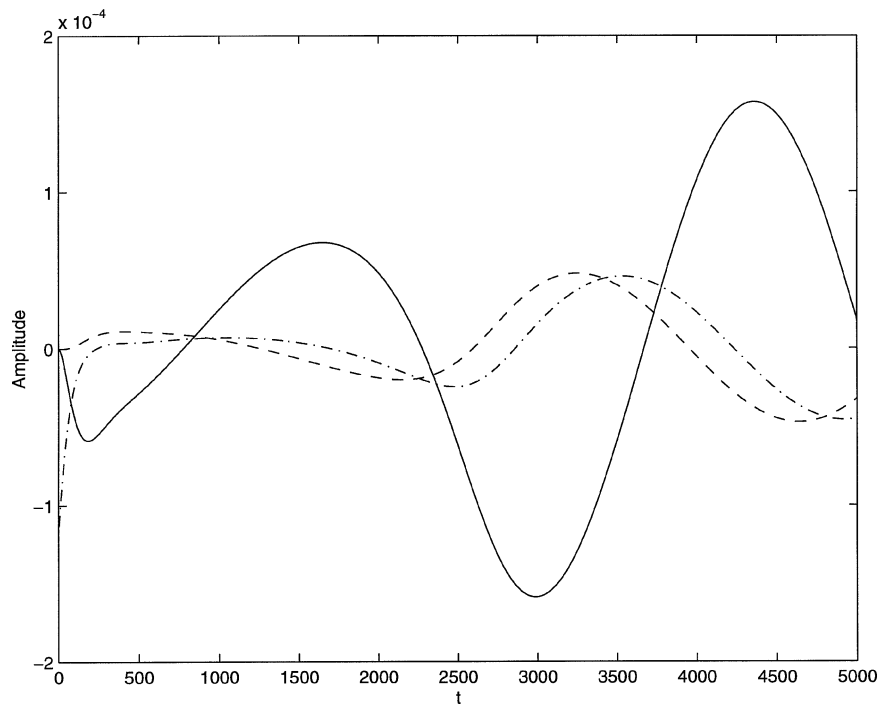
theory. The results show that a self-sustained disturbance amplification occurs. Concerning the feedback and the importance of different types of nonlinear terms, tests with the bb -terms switched off in the a -equations (13a)–(13d) have been performed. In all such tests the disturbance decays after the transient phase and no sustained state is established for the same parameter values that give sustained amplification if the bb -terms are included. Also, of course, if the linear forcing terms (ℓ_1 and ℓ_2) in (13e)–(13h) that cause the transient growth of the b -terms are switched off, no amplification or sustained state occur. In figures 4a–c the nonlinear bb -term (solid), ab -term (dashed) and aa -term (dashed dotted) from Eq. (13c) are presented. The parameter values are $R = 4000$, $\alpha = 0.03$, $\varepsilon = 7.5 \cdot 10^{-3}$ (4a), $\varepsilon = 7.75 \cdot 10^{-3}$ (4b) and $\varepsilon = 8.0 \cdot 10^{-3}$ (4c). The values of ε are here just below the threshold amplitude in 4a and 4b and just above the threshold amplitude in 4c. In all three cases the traces of the aa -terms (dashed dotted) and ab -terms (dashed) are quite similar until $t \approx 2500$. Concerning the bb -term (solid), the first negative period is very similar in the three cases but in the subsequent positive period the peak amplitude becomes approximately twice as large each time ε is increased. After the positive phase of larger amplitude of the bb -term in figure 4c, also the aa - and ab -terms change behaviour and start to oscillate. Roughly speaking, one can say that when the bb -term goes positive, the aa -term and ab -term tend to go negative and vice versa. The nonlinear terms of a_{2i} exhibit a similar behaviour. For a_{1r} and a_{1i} the situation is different and apart from the sign, the magnitude of the nonlinear terms are more equal. The conclusion from figures 4a–c is that close to the threshold amplitude, the significant changes of the nonlinear terms in (13c) are related to the bb -term for which the positive peak amplitude increases substantially when ε is increased in the neighbourhood of the threshold amplitude. There thus exists a nonlinear feedback from the transiently amplified disturbance Ω to the forcing disturbance Φ . Above a threshold of the initial disturbance amplitude, the feedback causes a sustained disturbance amplification continuing for all times.

In figure 3b the threshold relation $\varepsilon \sim R^{-3}$ is indicated. The obtained exponent $g = -3$ can be understood by a heuristic consideration of Eq. (13) similar to the one done by Baggett and Trefethen [25]. If the bb -term is





(b)



(c)

Figure 4. The nonlinear terms of Eq. (13c) for $R = 4000$, $\alpha = 0.03$. (a) $\varepsilon = 7.5 \cdot 10^{-3}$, (b) $\varepsilon = 7.75 \cdot 10^{-3}$, (c) $\varepsilon = 8.0 \cdot 10^{-3}$. The curves represent aa -term (dashed dotted), ab -term (dashed) and bb -term (solid).

assumed to be important at the stage when the disturbance becomes sustained, a simple heuristic model of the nonlinear development of, for example, a_{2r} in (13c) owing to the $b_{1r}b_{1i}$ -term is

$$\frac{da_{2r}}{dt} + \frac{j_{3,1}^2}{R}a_{2r} \approx 2\xi_5 R^2 \alpha^2 b_{1r}b_{1i}, \quad a_{2r}(0) = 0 \quad (15)$$

where ξ_5 is independent of α and R as explained earlier. If b_{1r} and b_{1i} are approximated by ε , the solution of (15) is

$$a_{2r}(t) \approx 2\xi_5 \frac{R^3 \alpha^2 \varepsilon^2}{j_{3,1}^2} (1 - e^{(-j_{3,1}^2/R)t}). \quad (16)$$

To achieve a sustained process, one can demand that $a_{2r} \approx \varepsilon$ after a long period of time. Thus (16) implies a threshold relation $\varepsilon \sim \alpha^{-2} R^{-3}$. (Another simple interpretation is just that the $b_{1r}b_{1i}$ -term should be able to overcome the damping term $((j_{3,1}^2/R)a_{2r})$. The heuristic analysis makes the numerically obtained value $g = -3$ reasonable. The exponent -3 is the same as obtained in several of the suggested low dimensional ad hoc models for transition which are compiled in [25]. From experimental results for pipe Poiseuille flow published by Darbyshire and Mullin [32], Baggett and Trefethen [25] suggested an exponent below -1 , perhaps in the vicinity of -1.5 . In relation to that value, the exponent indicated by the model (13) is then too low. It is reasonable to ascribe the discrepancy to the simplicity of the model exploited here. The model consists solely of small α components which are the most transiently amplified ones according to linear theory. The model therefore favours the transient part of the development. In reality a broad range of components are of course involved in the transition process. The generation of higher harmonics and more general structures are absent in a low-dimensional model like (13). An improved model containing more components which are less transiently amplified than $\alpha = 0$ components should possibly change the behaviour but also grow in complexity and become less tractable.

Although the present model is a low-dimensional one of streamwise elongated disturbances and should just be considered as a contribution to the debate, the obtained results indicate that the transient growth mechanism in combination with nonlinear feedback is capable of causing a state of permanent amplification of disturbances in pipe Poiseuille flow. The model is derived from the basic equations of a real flow case and incorporates the initial growth mechanism for small disturbances on a parabolic mean flow as well as the establishment of a self-sustained process. Moreover, although the full nonlinear problem and the transition phase are high-dimensional and include a broad range of wave components and modes, the basic self-sustaining process may essentially be related to low wave number components since they are the most transiently amplified and less damped ones.

Appendix

The coefficients of the system (13) have the following numerical values:

$$j_{1,1} = 3.83171, \quad j_{2,1} = 5.13562, \quad j_{3,1} = 6.38016,$$

$$\ell_1 = -0.99294, \quad \ell_2 = -0.24283,$$

$$\begin{aligned} \xi_1 &= 0.04752, & \xi_2 &= -0.00793, & \xi_3 &= 0.00172, & \xi_4 &= -0.42386, & \xi_5 &= 0.02942, \\ \xi_6 &= 0.00275, & \xi_7 &= -0.92344, & \xi_8 &= -2.44224, & \xi_9 &= -0.00038, & \xi_{10} &= -0.21485. \end{aligned}$$

Except for ξ_1 and ξ_5 which are independent of α , the values of the ξ 's are for $\alpha = 0.03$. Observe that the coefficients $\xi_1, \xi_2, \xi_3, \xi_5$ and ξ_6 of the nonlinear terms containing b in the a -equations (13a)–(13d) are closer to zero than the coefficients ξ_4 and ξ_7 of the nonlinear terms containing solely a terms in (13a)–(13d). However, since the nonlinear terms containing b in (13a–d) are multiplied by $\alpha^2 R^2$ or R , they have potential to reach the same magnitudes as the nonlinear terms containing solely a 's. This is substantiated in Section 4 and in figures 4a–c.

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